The Galaxy Formation Paradigm
When did galaxies form: a rough estimate -1

Consider a small \((\delta \rho / \rho << 1)\) spherical perturbation of radius \(r\):

\[
M(r) \approx \frac{4\pi}{3} r^3 \bar{\rho}_{\text{matter}} = \frac{4\pi}{3} r^3 \left(\frac{3H^2}{8\pi G}\right) \Omega_{\text{matter}}
\]

For \(H_0=100h \text{ km/s/Mpc}\),

\[
M(r) / M_{\text{sun}} \approx 1.16 \times 10^{12} h^2 \Omega_{\text{matter}} r_{\text{Mpc}}^3
\]

For \(h=0.7\) and \(\Omega_{\text{matter}}=0.25\),

\[
M(r) / M_{\text{sun}} \approx 1.4 \times 10^{11} r_{\text{Mpc}}^3
\]

i.e. an \(\sim L^*\) galaxy coalesced from matter within a comoving volume of roughly \(\sim 1-1.5 \text{ Mpc}\) radius
The typical radius of the visible parts of an L* galaxy as z=0 is 10-20 kpc.

However, we know that is embedded in a DM halo 5-10 times larger, say 75 kpc.

- The density enhancement represented by such a galaxy is thus

$$\delta = \frac{\delta \rho}{\rho_{\text{matter}}} \sim \left(\frac{1500}{75}\right)^3 \sim 8000$$

Since $\rho_{\text{matter}}$ evolves like $(1+z)^3$, the density fluctuation was of order $\delta \sim 1$ (“epoch of formation”) when $(1+z)^3 \sim 8000 (*)$, i.e.

- an L* galaxy could not have separated from Hubble flow before $z \sim 20$

(*) If we apply what we’ll learn later in the analysis of the spherical infall model, such a perturbation would “turn around” at a $z \sim 10$
Consider a 3-D fluid of density, pressure and velocity field and potential field given by:

\[ \rho(\vec{r},t), p(\vec{r},t), \vec{v}(\vec{r},t), \Phi(\vec{r},t) \]

which obey

**Continuity Eqn.:**

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \]

**Euler’s Eqn.:**

\[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho} \nabla p - \nabla \Phi \]

**Poisson’s Eqn.:**

\[ \nabla^2 \Phi = 4\pi G \rho \]

**Eqn. of State:**

\[ p(\vec{r}/t) = p[\rho(\vec{r},t)] \]

Gas dynamics eqns. are written in Eulerian form. In Cosmology, it is preferred to write them in Lagrangian form, where derivatives refer to changes in a gas parcel as it moves with velocity \( \vec{v} \):

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \]
Let the solution be a combination of an unperturbed or background component:

\[ \rho_0(\vec{r},t), p_0(\vec{r},t), \vec{v}_0(\vec{r},t), \Phi_0(\vec{r},t) \]

plus a small perturbation to \( \rho \)
(and similar ones for the other variables):

\[ \rho(\vec{r},t) = \rho_0(\vec{r},t) + \delta \rho(\vec{r},t) \]

In an expanding Universe, it is convenient to use comoving coordinates \( r \), so that

\[ \vec{x} = a(t) \vec{r} \]

where \( a(t) \) is the cosmological scale factor

After some algebra, the equation for the evolution of the perturbation is:

\[
\frac{d^2 \delta}{dt^2} + 2 \frac{\dot{a}}{a} \frac{d \delta}{dt} = \frac{c_s^2}{a^2} \nabla_c^2 \delta + 4\pi G \rho_0 \delta
\]

Where \( \delta = \delta \rho / \rho \), \( \nabla_c = a \nabla \) and \( c_s \) the speed of sound.
The solution of the perturbation equation can be expressed as a superposition of plane waves:

\[ \delta = C(\mathbf{k}_c)e^{i(\mathbf{k}_c \cdot \mathbf{r} - \omega(k_c)t)} \]

which yields:

\[ \frac{d^2 \delta}{dt^2} + 2 \frac{\dot{a}}{a} \frac{d \delta}{dt} = \delta(4\pi G \rho_o - k_c^2 c_s^2) \]

where “c” subscripts imply quantities expressed in the comoving reference frame.

First, we consider solutions in a static medium.
In a **static medium**,
\[ a \equiv 1, \quad \dot{a} = 0, \quad k = k_c \]

Time-evolving solutions are allowed; a *dispersion relation* is obtained:

\[ \omega^2 = c_s^2 k^2 - 4\pi G \rho_o \]

where \( k = |k| \).

The perturbation is a *sound wave* of
\[ \omega^2 \approx c_s^2 k^2 > 0 \]

when
\[ c_s^2 k^2 \gg 4\pi G \rho_o \]

i.e. when either the mean density is very low or when the wavelength of the perturbation is small.

\[ \lambda = \frac{2\pi}{k} \]
The value of $k$ for which $\omega=0$ is the Jeans’ wavenumber and the corresponding wavelength is **Jeans’ length**

\[ \lambda_J = \frac{2\pi}{k_J} = c_s \left( \frac{\pi}{G\rho_o} \right)^{1/2} \]

and **Jeans’ Mass**

\[ M_J = \frac{4\pi\rho_o}{3} \left( \lambda_J / 2 \right)^3 \]

- For $\lambda < \lambda_J$, $\omega$ is real and the solution is a sound wave
  - pressure provides support against gravitational collapse
- For $\lambda > \lambda_J$, $\omega$ is imaginary and the solution evolves exponentially, with a timescale ($\sim$ the same for all scales $\lambda \gg \lambda_J$)

\[ \tau = \frac{1}{|\omega|} = (4\pi G\rho_o)^{-1/2} \left[ 1 - \left( \frac{\lambda_J}{\lambda} \right)^2 \right]^{-1/2} \]

Which for $\lambda \gg \lambda_J$ is the free-fall collapse time

\[ \tau_{ff} \approx (G\rho_o)^{-1/2} \]
Interlude: Cosmic Evolution

Each cosmic component is characterized by an equation of state

\[ p = w \rho c^2 \]

and each evolves with the scale factor \( a(t) \)

In a flat Universe, the scale factor evolves as

\[ a(t) \propto t^{2/[3(1+w)]} \]

In a matter-dominated Universe (\( w \sim 0 \))

\[ a(t) \propto t^{2/3} \quad \text{and} \quad \rho_{\text{matter}} \propto a^{-3}(t) \propto (1+z)^3 \]

In a radiation-dominated Universe (\( w = 1/3 \))

\[ a(t) \propto t^{1/2} \quad \text{and} \quad \rho_{\text{rel}} \propto a^{-4}(t) \propto (1+z)^4 \]

In a \( \Lambda \)-dominated Universe (\( w = -1 \))

\[ a(t) \propto e^{Ht} \quad \text{and} \quad \rho_\Lambda = \text{const} \]

\[ 1 + z_{eq} = 4.3 \times 10^4 \Omega_{\text{matter}} h^2 / K_0 \]

\( K_0 = 1.68 \) for 3 neutrinos

(that's a \( z_{eq} \sim 4000 \))
The perturbation equation allows evolving solutions, a dispersion equation and the definition of a Jeans’ length, both in the case of a flat, matter dominated and a radiation dominated Universe.

The main difference between these solutions and those for a static medium is that the growth rate is not exponential.

In a flat Universe two evolving, power law solutions are allowed for $\lambda > \lambda_j$:

- a growing mode
  
  \[ \delta_+ \propto t^{2/3} \propto a \]
  
  \[ \delta_- \propto t^{-1} \]

- a decaying mode
  
  \[ \delta_+ \propto t \propto a^2 \]
  
  \[ \delta_- \propto t^{-1} \]

In the evolution of perturbations in an expanding Universe, two agents counteract gravity:

- pressure (as in the static case)
- universal expansion itself: fluctuation will not grow if collapse time is greater than Hubble time
In the evolution of perturbations in an expanding Universe, two agents counteract gravity:

- pressure (as in the static case)
- universal expansion itself: fluctuation will not grow if collapse time is greater than Hubble time

Consider a perturbation in a cosmic component which is NOT the dynamically dominant one, e.g. a mass perturbation during the radiation era.

If the perturbation wavelength is > the Jeans wavelength, pressure cannot prevent collapse, which would take place on a timescale

\[ \approx (G\rho_{\text{matter}})^{-1/2} \]

The expansion of the Universe does take place on a timescale which is smaller than the perturbation’s collapse timescale

\[ \approx (G\rho_{\text{radiation}})^{-1/2} \]

The Universe expands too fast for the perturbation to condense, i.e. even if \( \lambda > \lambda_J \), the perturbation does not grow.

A corollary of that conclusion is that all perturbations with scale smaller than the Hubble radius will not grow during the radiation era.
\begin{itemize}
\item In the radiation dominated era, only modes with $\lambda > r_{H}$ will grow in amplitude, and they will do so as $a^2$.
\item In the matter era, all modes with $\lambda > \lambda_J$ will grow in amplitude, and they will do so as the scale factor $a$.
\end{itemize}

Note that perturbations being referred to here are in the linear regime, i.e. $\delta\rho/\rho \ll 1$. Thus the proper size of the perturbation scales as $a$, the scale factor.

Also note that, while “pressure” in a baryonic gas is provided by collisions, in the DM component collisions are negligible; in that case, pressure arises from readjustment of orbits, and the speed of sound in the expression of Jeans’ length is replaced by the velocity dispersion of DM particles.
In a Universe with multiple components, the evolution of a perturbation depends on the type of component that is *perturbed* and on the type of component that is *gravitationally dominant*.

- In the expression for **Jeans’ length** \( \lambda_J \), the speed of sound (or the velocity dispersion) is that of the perturbed component; however, the density is that of the gravitationally dominant component.

\[
\lambda_J = \frac{2\pi}{k_J} = c_s \left( \frac{\pi}{G\rho_o} \right)^{1/2}
\]

Consider a DM perturbation.
A. Evolution of the density perturbation

1. $\lambda > r_H$, i.e. $a < a_{\text{enter}}$  
   perturbation scale > than Hubble radius  
   $$\delta \propto a^2$$

2. Perturbation enters horizon, radiation dominates:  
   $a_{\text{enter}} < a < a_{\text{eq}}$  
   $$\delta \propto \text{const}$$

3. Matter dominates:  
   $a_{\text{eq}} < a$  
   $$\delta \propto a$$

Log a
**Evolution of Jeans’ Length**

For DM, $c_s$ is replaced by the velocity dispersion of the DM particles. We distinguish three epochs:

1. **Before $a_{nr}$**, DM particles are relativistic and, while radiation is the dominant gravitational form:
   \[
   \lambda_J = \frac{2\pi}{k_J} = c_s \left( \frac{\pi}{G\rho_o} \right)^{1/2}
   \]

   \[\lambda_J \propto (G\rho_{\text{rad}})^{-1/2} \propto (a^{-4})^{-1/2} \propto a^2\]

2. **Between $a_{nr}$ and $a_{eq}$**, the velocity dispersion of DM particles decays as $a^{-1}$, so:
   \[\lambda_J \propto v(G\rho_{\text{rad}})^{-1/2} \propto a^{-1}(a^{-4})^{-1/2} \propto a\]

3. **After $a_{eq}$**, matter becomes the gravitationally dominant form:
   \[\lambda_J \propto v(G\rho_{\text{DM}})^{-1/2} \propto a^{-1}(a^{-3})^{-1/2} \propto a^{1/2}\]
Evolution of Jeans’ Mass

For DM, $c_s$ is replaced by the velocity dispersion of the DM particles. We distinguish three epochs:

1. Before $a_{nr}$, DM particles are relativistic and, while radiation is the dominant gravitational form ➔

2. Between $a_{nr}$ and $a_{eq}$, the velocity dispersion of DM particles decays as $a^{-1}$, so ➔

3. After $a_{eq}$, matter becomes the gravitationally dominant form ➔

$$M_J = \frac{4\pi \rho_o}{3} \left(\frac{\lambda_J}{2}\right)^3$$

$$M_J \propto (\rho_{DM}) \lambda_J^3 \propto a^{-3} a^6 \propto a^3$$

$$M_J \propto (\rho_{DM}) \lambda_J^3 \propto a^{-3} a^3 \propto \text{const}$$

$$M_J \propto (\rho_{DM}) \lambda_J^3 \propto a^{-3} a^{3/2} \propto a^{-3/2}$$
Gravitational Instability in an Expanding Universe - 8

Evolution of Dark Matter Perturbations

\[ a_{nr} \quad a_{eq} \]

\[ \lambda_J \quad M_J \]
Proceeding through analogous scaling relations

\[ a < a_{eq} \quad \rightarrow \quad \lambda_J \propto a^2, \quad M_J \propto a^3 \]
\[ a_{eq} < a < a_{rec} \quad \rightarrow \quad \lambda_J \propto a^{3/2}, \quad M_J \propto a^{3/2} \]
\[ a_{rec} < a \quad \rightarrow \quad \lambda_J \propto a^{1/2}, \quad M_J \propto a^{-3/2} \]

With one major difference wrt DM:
- Before the epoch of recombination, radiation and baryonic matter are tightly coupled, radiation providing the pressure of the coupled fluid.
- After recombination, the pressure drops by many orders of magnitude to practically zero (that of a few 1000K baryon fluid).
- As a result, both the Jeans’ Length and Mass drop precipitously. Numerically:

For:

\[ a < a_{eq} \]
\[ a_{eq} < a < a_{rec} \]
\[ a > a_{eq} \]
Gravitational Instability in an Expanding Universe - 11

1. Before entering the horizon, all super-horizon perturbations grow like $a^2$, as they exceed the Jeans’ length – which is ~ the Hubble radius.

2. After entering the horizon and for as long as radiation is the dominant gravitational form ($a < a_{eq}$), the perturbation is prevented from growing by the cosmic expansion.

3. At $a=a_{eq}$, DM becomes the gravitationally dominant form and the DM component of the perturbation resumes growth, its amplitude increasing like $a$.

4. The baryonic component of the perturbation is prevented from growing by the tight coupling with radiation, via Thomson scattering, until $a=a_{rec}$.

5. After recombination, the baryonic component can resume growth; between $a_{eq}$ and $a_{rec}$, however, the DM component of the perturbation has grown by a factor $a_{rec} / a_{eq} \sim 21\Omega_{\text{mass}} h^2$.

6. When baryons decouple from photons, they fall into the deep potential wells created by DM.

7. When $\delta_{\text{baryon}} = \delta_{\text{DM}}$, both components of the perturbation grow like $a$. 
Evolution of Baryonic Matter Perturbations

Solid = baryons
Dotted = DM

$\lambda_J$
$M_J$

$a_{nr}$ $a_{eq}$ $a_{rec}$
The description of the evolution of perturbations given so far only applies to small amplitudes, i.e. to the *linear regime*.

The next step is the analysis of the *Spherical Infall Model*.

To summarize (see notes for details and derivations):
- A slightly overdense region initially expands, but at a slightly lower rate than the universal rate.
- The expansion of each shell of the overdense region eventually stops and the motion “turns around” initiating collapse.
- The collapse is followed by a process of *violent relaxation* that reshuffles, randomizing, the orbits of the infalling particles, leading to *virialization*.
- Through dissipative processes, baryons lose energy and fall deeper in the potential well of DM.
- If the cooling time of the baryon gas is smaller than the collapse time, fragmentation will take place and smaller units can collapse.
A couple of important results of the spherical infall model analysis:

1. A perturbation of amplitude $\delta_i$ at redshift $z_i$ will turn around at

E.g. an overdensity of 1% at $z=1000$ will turn around at $z \approx 4.7$ at which time will be overdense by a factor of 4.6 wrt the background

2. The turn around overdensity is always 4.6 times the background density, independent on initial conditions

\[
1 + z_{\text{turn}} \approx 0.57(1 + z_i)\delta_i \approx \frac{\delta_o}{1.062}
\]

\[
1 + \delta_{\text{turn}} = \frac{9\pi^2}{16} \approx 5.6
\]
We define a virialized system as one where $E=U+K=-K$, i.e. $|U|=2K$

At turn around time, all the energy is in $U$, so that $E=U\sim 3GM^2 / 5R_{\text{max}}$

We define virial velocity, virial radius:

\[ |U| = \frac{3GM^2}{5R_{\text{vir}}} \quad \quad R_{\text{vir}} = \frac{R_{\text{turn}}}{2} \quad \quad v_{\text{vir}} = \left( \frac{6GM}{5R_{\text{turn}}} \right)^{1/2} \]

and the redshift of virialization

\[ 1 + z_{\text{vir}} = 0.36(1 + z_i) \delta_i = 0.63(1 + z_{\text{turn}}) \]

... as for the density at the epoch of virialization:

\[ \rho_{\text{vir}} \approx 2^3 \rho_{\text{turn}} \approx 8 \times 5.6 \times \rho_{bg}(z_{\text{turn}}) \approx 8 \times 56 \times \left( \frac{1 + z_{\text{turn}}}{1 + z_{\text{vir}}} \right)^3 \rho(z_{\text{vir}}) \]

\[ \approx 180 \rho_o (1 + z_{\text{vir}})^3 \]
The spherical infall model also yields scaling relations between size, mass and velocity that precur the observed relations we all use and love:

\[ R_{\text{vir}} = (164 \text{ kpc})(\Omega_{\text{matter}}h^2)^{-1/3} \left( \frac{M}{10^{12} M_{\odot}} \right)^{1/3} (1 + z_{\text{vir}})^{-1} \]

\[ V_{\text{vir}} = (125 \text{ km/s})(\Omega_{\text{matter}}h^2)^{1/6} \left( \frac{M}{10^{12} M_{\odot}} \right)^{1/3} (1 + z_{\text{vir}})^{1/2} \]

We can also define a \textit{Virial Temperature} via

\[ (3/2) \mu T_{\text{virial}} = (1/2)V_{\text{vir}}^2 \]

... yielding:

\[ T_{\text{vir}} = (3.68 \times 10^5 K)(\Omega_{\text{matter}}h^2)^{1/3} \left( \frac{M}{10^{12} M_{\odot}} \right)^{2/3} (1 + z_{\text{vir}}) \]
The cooling function of a primordial baryonic fluid, i.e.

$$\Lambda(T) \rightarrow \text{energy loss by radiation p.u. time p.u. volume}$$

is a function of $T$ and contributed to by 3 main processes:

1. Compton cooling with the CMB photons (important only at $z>10$)
2. Thermal bremsstrahlung (free-free) at $T > 10^6$ K
3. Bound-bound and recombination transitions of H and He at $10^4$ and $10^5$ K

[If elements > He are present, cooling rates below $10^6$ K increase dramatically. We define **cooling time**

$$t_{cool} = \frac{E}{|dE/dt|} = \frac{3n_b kT}{n_b^2 \Lambda(T)}$$

which is useful to compare to the Gravitational collapse time

$$t_{dyn} \approx (G\rho)^{-1/2} \propto n_b$$

By setting $t_{cool} = t_{dyn}$, we obtain a relationship between density and temperature.
Fig. 16.2. The cooling rate per unit volume $\Lambda(T)$ of an astrophysical plasma of number density 1 nucleus cm$^{-3}$ by radiation for different cosmic abundances of the heavy elements ranging from zero metals to the present abundance of the heavy elements as a function of temperature $T$ (Silk and Wyse 1993, after Sutherland and Dopita 1993). In the zero metal case, the two maxima of the cooling curve are associated with the recombination of hydrogen ions and doubly ionised helium (see also Sect. 19.5 and Fig. 19.3).
$1D \sigma_v = \left[\frac{kT}{\mu m_p}\right]^{1/2}/\text{km s}^{-1}$
Aside: gravitational instability in a rotating disk
In a **static medium**, \( a \equiv 1, \dot{a} = 0, k = k_c \)

Time-evolving solutions are allowed; a **dispersion relation** is obtained:

\[
\omega^2 = c_s^2 k^2 - 4\pi G \rho_o
\]

The perturbation is a **sound wave** of \( \omega^2 \cong c_s^2 k^2 > 0 \) when \( c_s^2 k^2 \gg 4\pi G \rho_o \)

i.e. when either the mean density is very low or when the wavelength of the perturbation \( \lambda = 2\pi / k \) is small.
When \( \omega^2 < 0 \), so that \( \gamma = \pm \sqrt{-\omega^2} \) is a real quantity, then the
temporal dependence of the perturbation is \( \propto e^{\pm \gamma t} \), which corresponds to an exponentially either growing or decaying solution.

The possible existence of a growing solution means that 

\[ \textit{the system is unstable} \]

\textit{Instability occurs if} \( k^2 < k_J^2 \equiv \frac{4 \pi G \rho_o}{v_s^2} \)

where \( k_J \textit{is the Jeans’ wavenumber} \).
**Jeans Instability**

**Jeans’ wavenumber**

\[ k^2 < k_j^2 \equiv 4\pi G \rho_o / c_s^2 \]

**Jeans’ wavelength**

\[ \lambda^2 > \lambda_j^2 \equiv \pi c_s^2 / G \rho_o \]

**Jeans’ Mass**

\[ M > M_j \equiv (4\pi / 3) \rho_o (\lambda_j / 2)^3 = (\pi / 6) \rho_o \left( \frac{\pi c_s^2}{G \rho_o} \right)^{3/2} \]

The characteristic timescale of the evolving perturbation is

\[ \tau = |\gamma|^{-1} = (4\pi G \rho_o)^{-1/2} \left[ 1 - (\lambda_j / \lambda)^2 \right]^{-1/2} \]

Which approximates the free-fall collapse time for \( \lambda >> \lambda_j \)

\[ \tau_{ff} \approx (G \rho_o)^{-1/2} \]
Consider now a disk of negligible thickness, surface density $\Sigma_0$, confined to the plane $z=0$ and rotating with constant angular speed $\Omega$

The problem is best treated in the rotating reference frame. The main change wrt the Jeans’ case is then the appearance of 2 additional terms in Euler’s equation, representing inertial forces (centrifugal and Coriolis):

$$\frac{\partial v}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\Sigma} - \nabla \Phi - 2\Omega \times \mathbf{v} + \Omega^2 (x\mathbf{e}_x + y\mathbf{e}_y)$$

After the usual algebra, we obtain the dispersion relation

$$\omega^2 = 4\Omega^2 - 2\pi G\Sigma_0 k + k^2 c_s^2$$

and instability arises if $\omega^2 < 0$

(The standard case dispersion relation was: $\omega^2 = c_s^2 k^2 - 4\pi G\rho_0$)
**Stability of a rotating, thin disk**

1. In the Jeans’ case:

\[ \omega^2 = c_s^2 k^2 - 4\pi G \rho_o \]

when either the pressure term becomes negligible or for large wavelengths, *all perturbations grow at \( \sim \) the same rate*

2. In the thin sheet case, if pressure is negligible and the sheet doesn’t rotate, *instability grows faster as wavelength decreases*:

\[ \omega^2 = 4\Omega^2 - 2\pi G \Sigma_o k + k^2 c_s^2 \]

\[ \tau = \gamma^{-1} = (2\pi G \Sigma_o k)^{-1/2} \]

3. Keep pressure term negligible, but allow for sheet rotation. In this case, only perturbations with

\[ k > 2\Omega^2 / \pi G \Sigma_o \]

are unstable. They grow at the rate

\[ \gamma = (2\pi G \Sigma_o k - 4\Omega^2)^{1/2} \]

*The rate increases as wavelength decreases (as case 2), but rotation inhibits growth of large perturbations*
4. Assume now the sheet is non-rotating, but the pressure term cannot be ignored. Then only perturbations with are unstable, i.e.

\[ k < \frac{2\pi G \Sigma_o}{c_s^2} \]

and they grow at the rate

\[ \gamma = \left( -c_s^2 k^2 + 2\pi G \Sigma_o k \right)^{1/2} \]

which increases with increasing wavelength:

pressure inhibits the growth of small-scale perturbations

While neither pressure nor rotation can fully stabilize the sheet, a combination of the two can do so.
**Stability of a rotating, thin disk**

Because the dispersion relation of a rotating sheet is quadratic in $k$, the wavenumber $k_m$ for which the growth rate maximizes is one of largest instability:

$$\omega^2 = 4\Omega^2 - 2\pi G \Sigma_o k + k^2 c_s^2$$

$$\frac{\partial (\gamma^2)}{\partial k} = 0 \Rightarrow -2\pi G \Sigma_o + 2k c_s^2 = 0 \Rightarrow k_m = \frac{\pi G \Sigma_o}{c_s^2}$$

For $k<k_m$, instability is inhibited by rotation.

For $k>k_m$, instability is inhibited by pressure.

If, for $k=k_m$, $\gamma^2 < 0$, then the sheet is stable for all $k$.

**Stability condition:**

$$4\Omega^2 - 2\pi G \Sigma_o k_m + k_m^2 c_s^2 > 0$$

a.k.a.  

$$\frac{2\Omega c_s}{\pi G \Sigma_o} > 1$$
Stability of a rotating, thin disk

The extension of the rotating disk stability to a differentially rotating disk is algebraically laborious; the resulting dispersion relation of a gas disk is:

$$(m\Omega - \omega)^2 = \kappa^2 - 2\pi G \Sigma_o |k| + k^2 c_s^2$$

...and for a stellar disk it is:

$$(m\Omega - \omega)^2 = \kappa^2 - 2\pi G \Sigma_o |k| F[(\omega - m\Omega)/\kappa, k^2 \sigma^2 / \kappa^2]$$

where $F$ is a tabulated function of stellar velocity dispersions in $R$, $\sigma$, $\kappa$ is the epicyclic frequency and other already defined variables.
The current azimuthal angle of the Sun on its epicycle is 
\[ \chi = \text{AGS} = 146^\circ \]

The size of the epicycle is 
\[ A_R = \text{AG} = 500 \text{ pc} \]
\[ 2A_R \frac{\Omega}{\kappa} = \text{BG} = 700 \text{ pc} \]

And the velocity components in \( x, y, z \) are 
\[ 9, 12, 7 \text{ km/s} \]

The motion of the Sun can be described as an elliptical epicycle in the plane of the disk and a simple harmonic in the \( z \)-direction:

\[
\begin{align*}
x &= R(t) - R_g = A_R \sin(\kappa t) \\
y &= \varphi_0 + \Omega t + \frac{2\Omega}{\kappa R_g} A_R \cos(\kappa t) \\
z &= A_z \sin(\nu t + \zeta)
\end{align*}
\]

Where \( \kappa \) is the epicyclic frequency related to Oort's constants via

\[
\kappa / \Omega = 2\left[ -B / (A - B) \right]^{1/2} \approx 1.3
\]
Stability of a rotating, thin disk

For $m=0$ (axisymmetric perturbations), the differentially rotating disk dispersion relation is

$$ \omega^2 = \kappa^2 - 2\pi G \Sigma_o |k| + k^2 c_s^2 $$

And the corresponding stability conditions, for gas and stars, are:

- $Q_g \equiv \frac{c_s \kappa}{\pi G \Sigma_o} > 1$
- $Q_* \equiv \frac{\sigma(R) \kappa}{3.36 G \Sigma_o} > 1$

**Toomre Stability Criteria**

Note:
- The Solar neighborhood, with $Q \sim 1.7$, appears stable
- Increasing stellar velocity dispersion, $\sigma$, or gas motions ($v_s$), increases stability
- Disks with high/low $Q$ are referred to as “hot”/“cold”
In the case of a spherical cloud rotating with angular velocity $\Omega$, the resulting dispersion relation is

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_o + 4\Omega^2 \sin^2 \theta$$

(Non-rotating case: $\omega^2 = c_s^2 k^2 - 4\pi G \rho_o$)

$\theta$ is the angle between an arbitrary direction and the rotation axis.

The growth rate of the instability is greatest for $\theta = 0$, i.e. for propagation parallel to the rotation axis. Thus the cloud tends to flatten into a disk.

$\Rightarrow$ Ambipolar diffusion
Note that the Jeans’ mass for a medium with the average properties of a GMC \((n \sim 10^8 \text{ m}^{-3}, T \sim 25K)\) is significantly > than 1 solar:

\[
M > M_J \equiv (4\pi / 3) \rho_o (\lambda_J / 2)^3 = (\pi / 6) \rho_o \left( \frac{\pi c_s^2}{G \rho_o} \right)^{3/2} \approx 3 \rho_o^{-1/2} \left( \frac{\mathcal{R}_* T}{G \mu} \right)^{3/2}
\]

with \(\mathcal{R}_* = 8.31 \times 10^3\), the ideal gas constant in MKS units and \(\mu\) the molecular weight \(\sim 1\), yields

\[
M_J \approx 650 M_{\text{sun}}
\]

However, GMCs have much denser cores with \(n \sim 5 \times 10^{11} \text{ m}^{-3}\), \(T \sim 10K\), yielding collapse of single stellar mass \(\Rightarrow\) fragmentation
Homework assignment: (due October 29)

Use the tabulation of HI properties of galaxies provided at the course website, to verify whether the assumption of optical thinness for the 21 cm line emission of galactic disks is justified. Is there a signal? Is it related to HI mass?
Extra slides (not used in class)
In a Universe with multiple components, the evolution of a perturbation depends on the type of component that is perturbed and on the type of component that is gravitationally dominant.

- In the expression for Jeans' length, the speed of sound (or the velocity dispersion) is that of the perturbed component; however, the density is that of the gravitationally dominant component.

\[ \lambda_J = \frac{2\pi}{k_J} = c_s \left( \frac{\pi}{G\rho_o} \right)^{1/2} \]

Consider, for example, a DM perturbation of scale \( \lambda \) which has entered the horizon, and is such that

\[ t_{\text{pressure}} \approx \frac{\lambda}{v} > \left( G\rho_{DM} \right)^{-1/2} \]

Pressure is unable to prevent collapse.

However, if the Universe is radiation dominated, the expansion time is determined by \( \rho_{\text{rad}} \); then the Universe expands too fast and prevents the DM perturbation from collapsing.

\[ t_{\exp} \approx \left( G\rho_{\text{rad}} \right)^{-1/2} < \left( G\rho_{DM} \right)^{-1/2} \]

**In the radiation dominated era, perturbations with scale < \( r_H \) cannot grow**
* Consider first the evolution of perturbations of scale > the Hubble radius $r_H$; 

$\Rightarrow$ Pressure doesn’t play a role on scales > $r_H$, thus the perturbation grows as $\delta_+$

i.e. as $\delta_+ \propto t^{2/3} \propto a$ in a matter dominated Universe

and as $\delta_+ \propto t \propto a^2$ in a radiation dominated one.

* We can understand that as follows:

- consider the perturbation as a slightly overdense sphere within a flat (k=0) Universe of scale factor $a_o$; matter outside the sphere does not affect evolution within the sphere, so the sphere evolves like a Friedman Universe with density higher than critical, i.e. k=1

- Let the density of the perturbation be $\rho_1$ and its scale factor $a_1$

* The evolution of the scale factor in the perturbed region and in the flat background are then respectively

$$H_1 + 1/a_1^2 = (8\pi G / 3) \rho_1$$

$$H_o = (8\pi G / 3) \rho_o$$
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* Let’s compare the densities when the expansion rates are equal \( H_1 = H_2 \)

\[
\frac{\rho_1 - \rho_o}{\rho_o} = \frac{\delta \rho}{\rho_o} = \frac{3}{8\pi G \rho_o a_1^2}
\]

* If the perturbation is very small, \( a_1 \sim a_o \)

- Remembering the evolution of the scale factor in
  - radiation dominated
  - and
  - matter dominated
  Universes

  i.e. we recover the behavior of the growing mode \( \delta_+ \)

\[ H_1 + 1/a_1^2 = (8\pi G / 3) \rho_1 \]
\[ H_o = (8\pi G / 3) \rho_o \]

\[ \delta \propto \rho_o a_1^2 \]

\[ \delta \propto a^2 \propto t \]
\[ \delta \propto a \propto t^{2/3} \]

The amplitude of a perturbation with \( \lambda > r_H \) always grows